Logarithmic Schrödinger equation and isothermal fluids

Rémi Carles

Part 1. Logarithmic Schrödinger equation

1. Linear equations

1.1. Heat equation.

$$\partial_t u = \frac{1}{2}\Delta u, \quad x \in \mathbb{R}^d, \quad u_{|t=0} = u_0 \in L^1(\mathbb{R}^d).$$

Explicit solution : $u(t,x) = \dfrac{1}{(2\pi t)^{d/2}} \displaystyle\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} u_0(y)dy.$

Large time description:

$$\hat{u}(t,\xi) = e^{-\frac{t}{2}|\xi|^2}\hat{u}_0(\xi) = \underbrace{e^{-\frac{t}{2}|\xi|^2}\hat{u}_0(0)}_{\text{order } t^{-d/(2p)} \text{ in } L^p} + \underbrace{e^{-\frac{t}{2}|\xi|^2}\left(\hat{u}_0(\xi) - \hat{u}_0(0)\right)}_{\mathcal{O}\left(|\xi|e^{-\frac{t}{2}|\xi|^2}\right)\text{: order } t^{-(d+1)/(2p)} \text{ in } L^p}$$

If $m := \int_{\mathbb{R}^d} u_0 \neq 0$,

$$(t,x) \underset{t\to\infty}{\sim} \frac{m}{(2\pi t)^{d/2}} e^{-|x|^2/(2t)}.$$

1.2. Schrödinger equation.

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \quad x \in \mathbb{R}^d, \quad u_{|t=0} = u_0 \in L^2(\mathbb{R}^d).$$

Explicit solution : $u(t,x) = \dfrac{1}{(2i\pi t)^{d/2}} \displaystyle\int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} u_0(y)dy.$

Two consequences:

- Dispersion: $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \dfrac{1}{|t|^{d/2}}\|u_0\|_{L^1(\mathbb{R}^d)}.$
- Large time description: $\|u(t) - A(t)u_0\|_{L^2(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0$, where

$$A(t)u_0(x) = \frac{1}{(it)^{d/2}}\hat{u}_0\left(\frac{x}{t}\right)e^{i\frac{|x|^2}{2t}}.$$

Universal oscillation, but the profile depends on the initial data.

Example 1.1 (Explicit computation in the Gaussian case).

$$\operatorname{Re} z > 0 : \quad e^{i\frac{t}{2}\Delta}\left(e^{-z\frac{|x|^2}{2}}\right) = \frac{1}{(1+itz)^{d/2}}e^{-\frac{z}{1+itz}\frac{|x|^2}{2}}.$$

2. Nonlinear Schrödinger equation

For $\lambda \in \mathbb{R}$, $0<\sigma<\frac{2}{(d-2)_+}$, consider:

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{2\sigma} u, \quad x\in\mathbb{R}^d, \quad u_{|t=0} = u_0 \in H^1(\mathbb{R}^d). \tag{2.1}$$

2.1. Invariants. Space and time translations.
Gauge.
Galilean: if $u(t,x)$ solve NLS, then for any $\mathbf{v}\in\mathbb{R}$, so does $u(t,x-\mathbf{v}t)e^{i\mathbf{v}\cdot x - i|\mathbf{v}|^2t/2}$.
Useful to construct multisolitons.
Formal conservations:

$$M(u(t)) = \|u(t)\|^2_{L^2(\mathbb{R}^d)},$$
$$J(u(t)) := \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx,$$
$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{\lambda}{\sigma+1}\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)}.$$

2.2. Defocusing case. If $\lambda>0$: global existence ($u\in L^\infty(\mathbb{R};H^1(\mathbb{R}^d))$), and if $\sigma>2/d$,

$$\exists u_+ \in H^1(\mathbb{R}^d), \quad \|u(t) - e^{i\frac{t}{2}\Delta}u_+\|_{H^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0.$$

The (inverse of) the wave operator is not trivial: $u_0\mapsto u_+$ is one-to-one.

2.3. Focusing case. If $\lambda<0$: finite time blow-up is possible when $\sigma\geqslant 2/d$,

$$\lim_{t\to T^*}\|\nabla u(t)\|_{L^2} = \infty.$$

For $\sigma\geqslant 2/d$, small data scattering.

Existence of large stationary solutions: $u(t,x) = e^{i\omega t}\psi(x)$, ψ ground state. Orbitally stable iff $\sigma<2/d$.

3. Logarithmic Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u, \quad u_{|t=0} = u_0, \tag{3.1}$$

with $x\in\mathbb{R}^d$, $d\geqslant 1$, and $\lambda\in\mathbb{R}$.

Introduced in [**5**] to satisfy the following tensorization property: if the initial datum is a tensor product,

$$u_0(x) = \prod_{j=1}^d u_{0j}(x_j),$$

then the solution to (3.1) is given by

$$u(t,x) = \prod_{j=1}^d u_j(t,x_j),$$

where each u_j solves a one-dimensional equation,

$$i\partial_t u_j + \frac{1}{2}\partial^2_{x_j} u_j = \lambda\ln\left(|u_j|^2\right)u_j, \quad u_{j|t=0} = u_{0j}.$$

The logarithmic nonlinearity is the only one satisfying such a property.

3.1. Invariants. Same as before, and

$$M(u(t)) = \|u(t)\|^2_{L^2(\mathbb{R}^d)},$$
$$J(u(t)) := \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx,$$
$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda\int_{\mathbb{R}^d} |u(t,x)|^2\left(\ln|u(t,x)|^2 - 1\right)dx.$$

Dispersive vs. nondispersive?

Size: If u solves (3.1), then for all $k \in \mathbb{C}$, so does

$$u_k(t,x) := ku(t,x)e^{it\lambda\ln|k|^2}.$$

This shows that the size of the initial data alters the dynamics only through a purely time dependent oscillation, a feature which is fairly unusual for a nonlinear equation. For $k > 0$,

$$\frac{d}{dk}u_k(t,x) = (1+2it)u(t,x)e^{it\lambda\ln|k|^2}.$$

No limit as $k \to 0$ for $t > 0$: the flow map $u_0 \mapsto u(t)$ cannot be C^1, whichever function spaces are considered for u_0 and $u(t)$, respectively; it is at most Lipschitzean.

4. Cauchy problem

$$W := \left\{u \in H^1(\mathbb{R}^d)\,,\ x \mapsto |u(x)|^2\ln|u(x)|^2 \in L^1(\mathbb{R}^d)\right\}.$$

THEOREM 4.1 ([**19**]). *$\lambda < 0$, $u_0 \in W$: unique, global solution $u \in C(\mathbb{R}; W)$. The mass $M(u)$ and the energy $E(u)$ are independent of time.*

- Proof by compactness arguments, using a regularization of the nonlinearity.
- Alternative proof by Masayuki Hayashi [**27**], proving the strong convergence of a sequence of approximate solutions.

Globalization: a priori estimates

$$0 \leqslant E_+(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda\int_{|u|<1}|u(t,x)|^2\ln|u(t,x)|^2dx$$
$$\leqslant E(u_0)\underbrace{-\lambda}_{+|\lambda|}\int_{|u|>1}|u(t,x)|^2\ln|u(t,x)|^2dx.$$

Since the logarithm grows slowly,

$$\int_{|u|>1}|u(t,x)|^2\ln|u(t,x)|^2dx \leqslant C_\varepsilon\int_{|u|>1}|u(t,x)|^{2+\varepsilon}dx$$
$$\lesssim \|u(t)\|^{2+\varepsilon-\varepsilon d/2}_{L^2(\mathbb{R}^d)}\|\nabla u(t)\|^{\varepsilon d/2}_{L^2(\mathbb{R}^d)},$$

provided that $\varepsilon < 2d/(d-2)_+$. Using the conservation of mass,

$$E_+(u(t)) \leqslant E(u_0) + C_\varepsilon E_+\left(u(t)\right)^{\varepsilon d/4}.$$

REMARK 4.2. Case $\lambda > 0$: globalization would require the control of

$$\int_{|u|<1}|u(t,x)|^2\ln\frac{1}{|u(t,x)|^2}dx \leqslant C_\varepsilon\int_{|u|<1}|u(t,x)|^{2-\varepsilon}dx.$$

Uniqueness:

LEMMA 4.3 ([**19**]). *We have*

$$\left|\operatorname{Im}\left(\left(z_2\ln|z_2|^2 - z_1\ln|z_1|^2\right)\left(\bar z_2-\bar z_1\right)\right)\right| \leqslant 4|z_2-z_1|^2\,,\quad \forall z_1,z_2\in\mathbb{C}\,.$$

+ show that actually, $u\in C(\mathbb{R};L^2)$.

Another compactness method:

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = \lambda\ln\left(\varepsilon+|u^\varepsilon|^2\right)u^\varepsilon,\quad u^\varepsilon_{|t=0}=u_0. \tag{4.1}$$

Assuming $u_0\in H^1$, $\langle x\rangle^\alpha u_0\in L^2$ for some $0<\alpha\leqslant 1$.

5. Special solutions

Important general property:

$$i\partial_t u + \frac{a(t)}{2}\partial_x^2 u = b(t)\frac{x^2}{2}u \quad ; \quad u_{|t=0}=u_0.$$

If u_0 is Gaussian, so is $u(t,\cdot)$.

5.1. General computation. Suppose $d=1$, and plug $u(t,x)=b(t)e^{-a(t)x^2/2}$ into (3.1):

$$i\dot b - i\dot a\frac{x^2}{2}b - \frac{ab}{2} + a^2\frac{x^2}{2}b = \lambda\left(\ln\left(|b|^2\right) - \left(\operatorname{Re}a\right)x^2\right)b,$$

hence

$$i\dot a - a^2 = 2\lambda\operatorname{Re}a\,;\quad i\dot b - \frac{ab}{2} = \lambda b\ln\left(|b|^2\right).$$

We can express b as a function of a:

$$b(t) = b_0\exp\left(-i\lambda t\ln\left(|b_0|^2\right) - \frac{i}{2}A(t) - i\lambda\operatorname{Im}\int_0^t A(s)sds\right),$$

where we have set $A(t):=\displaystyle\int_0^t a(s)ds$. So we focus on

$$i\dot a - a^2 = 2\lambda\operatorname{Re}a,\quad a_{|t=0}=a_0=\alpha_0+i\beta_0.$$

We seek a of the form $a=-i\dfrac{\dot\omega}{\omega}$.

We get: $\ddot\omega = 2\lambda\omega\operatorname{Im}\dfrac{\dot\omega}{\omega}$.

Polar decomposition: $\omega=re^{i\theta}$,

$$\ddot r - (\dot\theta)^2 r = 2\lambda r\dot\theta\,;\quad \ddot\theta r + 2\dot\theta\dot r = 0.$$

Notice that

$$\dot\theta_{|t=0}=\alpha_0\,,\quad \left(\frac{\dot r}{r}\right)_{|t=0} = -\beta_0\,.$$

We decide $r(0)=1$ so $\dot\theta(0)=\operatorname{Re}a_0=\alpha_0$ and $\dot r(0)=-\operatorname{Im}a_0=-\beta_0$. Note

$$\frac{d}{dt}\left(r^2\dot\theta\right) = r\left(2\dot r\dot\theta + r\ddot\theta\right) = 0\,,$$

and we can express the problem in terms of r only:

$$a(t)=\frac{\alpha_0}{r(t)^2} - i\frac{\dot r(t)}{r(t)},\quad \ddot r = \frac{\alpha_0^2}{r^3}+2\lambda\frac{\alpha_0}{r},\quad r(0)=1\,,\ \dot r(0)=-\beta_0\,.$$

Multiply by $\dot{r}$ and integrate:

$$(\dot{r})^2 = \beta_0^2 + \alpha_0^2 - \frac{\alpha_0^2}{r^2} + 4\lambda\alpha_0 \ln |r|. \tag{5.1}$$

Cauchy-Lipschitz: local solution.
Supposing $r \to 0$ leads to a contradiction:

$$r(t) \geqslant \delta > 0, \quad t \geqslant 0.$$

5.2. Nondispersive case: $\lambda < 0$. In view of (5.1), r is bounded.
Every solution is periodic in time: *breather*.
Particular case: $\beta_0 = 0$, $\alpha_0 = -2\lambda$ implies $r \equiv 1$.

$$u_\omega(t,x) = e^{i\omega t} e^{\frac{d}{2} - \frac{\omega}{2\lambda}} e^{\lambda |x|^2}.$$

Soliton, for all $\omega \in \mathbb{R}$. Known as *Gausson* [**6**].

5.3. Dispersive case: $\lambda > 0$. We can prove: for $t \geqslant T$, $\ddot{r} > 0$, and $r(t) \to \infty$ as $t \to \infty$. Hence

$$\ddot{r}_{\text{eff}} = \frac{2\lambda\alpha_0}{r_{\text{eff}}} \quad (\alpha_0 > 0).$$

Up to scaling (and initial data): $\ddot{\tau} = \dfrac{2\lambda}{\tau}$.
By integration,

$$\dot{r}_{\text{eff}} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\text{eff}}},$$

$$\dot{r}_{\text{eff}} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\text{eff}}}.$$

Separate variables:

$$\int^{r_{\text{eff}}} \frac{dz}{\sqrt{C_0 + 4\lambda\alpha_0 \ln z}} = t - T.$$

Set $y = \sqrt{C_0 + 4\lambda\alpha_0 \ln z}$. The left hand side becomes

$$\frac{1}{2\lambda\alpha_0} \int^Y e^{(y^2 - C_0)/(4\lambda\alpha_0)} dy.$$

Dawson function:

$$\int^x e^{y^2} dy \underset{x\to\infty}{\sim} \frac{1}{2x} e^{x^2} \Longrightarrow \frac{r_{\text{eff}}}{\sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\text{eff}}}} \underset{t\to\infty}{\sim} t.$$

Since $r_{\text{eff}} \to \infty$, $\dfrac{r_{\text{eff}}}{\sqrt{4\lambda\alpha_0 \ln r_{\text{eff}}}} \underset{t\to\infty}{\sim} t$, hence

$$r_{\text{eff}}(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda\alpha_0 \ln t}.$$

Crucial remark: C_0 has disappeared (at leading order). All the Gaussian solutions have the same asymptotic profile, with a nonstandard dispersion.

6. Solitons

LEMMA 6.1 ([**17**]). *Let $\lambda < 0$ and $k < \infty$ such that*

$$L_k := \{u \in W,\ \|u\|_{L^2(\mathbb{R}^d)} = 1,\ E(u) \leqslant k\} \neq \emptyset.$$

Then $\inf\limits_{\substack{u \in L_k \\ 1 \leqslant p \leqslant \infty}} \|u\|_{L^p(\mathbb{R})} > 0.$

No solution is dispersive in the case $\lambda < 0$.

6.1. Gaussons. Solutions

$$u_\omega(t,x) = e^{i\omega t} e^{\frac{d}{2}-\frac{\omega}{2\lambda}} e^{\lambda |x|^2}$$

are *orbitally stable*: [**17**] for the radial case, [**2**] for the general case (proof based on the logarithmic Sobolev inequality).

THEOREM 6.2 ([**17, 2**]). *Let $\lambda < 0$ and $\omega \in \mathbb{R}$. Set*

$$\phi_\omega(x) = e^{\frac{d}{2}-\frac{\omega}{2\lambda}} e^{\lambda |x|^2}.$$

For any $\varepsilon > 0$, there exists $\eta > 0$ such that if $u_0 \in W$ satisfies $\|u_0 - \phi_\omega\|_X < \eta$, then the solution u to (3.1) *exists for all $t \in \mathbb{R}$, and*

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^d} \|u(t) - e^{i\theta}\phi_\omega(\cdot - y)\|_W < \varepsilon.$$

6.2. Multigaussons. Superposition principle observed numerically in [**4**], and proved in [**22**]: starting from finitely many initial Gaussians distant of $\approx \varepsilon$, the solution of is well approximated by the sum of the corresponding solutions, over a time $o(\varepsilon^{-2})$ (the error is of order $e^{c_0 t - c_1/\varepsilon^2}$ for some $c_0, c_1 > 0$ expressed explicitly in [**22**]).

Multigausson: using the Galilean invariance, introduce, for some $k \geqslant 1$

$$G_k = \sum_{j=1}^k \Gamma_j(t,x), \quad \mathbb{B}_k = \sum_{j=1}^k B_j(t,x)$$

where the Γ_j's are Gaussons associated with pairwise different velocities $\mathbf{v}_j$, and the B_j's are (more general) breathers associated with pairwise different velocities $\mathbf{v}_j$.

THEOREM 6.3 ([**24**]). *Let $d \geqslant 1$ and $\lambda < 0$.*

- *Multibreathers: there exists a solution $u \in C_b(\mathbb{R}; W)$ to* (3.1)*, $c, C > 0$ such that*
$$\|u(t) - \mathbb{B}_k(t)\|_{L^2(\mathbb{R}^d)} \leqslant Ce^{-ct^2}.$$
- *Multigaussons: there exists a solution $u \in C_b(\mathbb{R}; \Sigma)$ to* (3.1)*, $c, C > 0$ such that*
$$\|u(t) - G_k(t)\|_\Sigma \leqslant Ce^{-ct^2}.$$

Comments:

- Method based on compactness techniques, as introduced in [**30**].
- The linearized operator around the Gausson seems to be nice (harmonic oscillator). However, the logarithm is singular at zero, and so linearizing becomes a delicate matter.
- Localized energy functionals involving a linearized functional, which is not the linearized energy.

7. Dispersive case

7.1. Main results.

THEOREM 7.1 ([**16**]). *Let $\lambda > 0$. For $u_0 \in \Sigma \setminus \{0\}$, (3.1) has a unique solution $u \in L^\infty_{\rm loc}(\mathbb{R};\Sigma))$. Introduce the solution $\tau \in C^\infty(\mathbb{R})$ to the ODE*

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0. \tag{7.1}$$

Then, as $t \to \infty$, $\tau(t) \sim 2t\sqrt{\lambda \ln t}$ and $\dot{\tau}(t) \sim 2\sqrt{\lambda \ln t}$. Introduce $\gamma(x) := e^{-|x|^2/2}$, and rescale the solution to $v = v(t,y)$ by setting

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}} \exp\left(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\right). \tag{7.2}$$

Then we have

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy,$$

and

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \textit{weakly in } L^1(\mathbb{R}^d).$$

COROLLARY 7.2. *Let $u_0 \in H^1 \cap \mathcal{F}(H^1) \setminus \{0\}$, and $0 < s \leqslant 1$. As $t \to \infty$,*

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot{H}^s(\mathbb{R}^d)$ denotes the standard homogeneous Sobolev space.

PROOF IN THE CASE $s = 1$.

$$\begin{aligned}\nabla u(t,x) &= \frac{1}{\tau(t)^{d/2}} \nabla_x \left(v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}} \right) \\ &= \underbrace{\frac{1}{\tau(t)} \frac{1}{\tau(t)^{d/2}} \nabla_y v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \frac{1}{\tau}\|\nabla v\|_{L^2} = \mathcal{O}(1).} + \underbrace{i\dot{\tau} \frac{1}{\tau(t)^{d/2}} \frac{x}{\tau} v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \dot{\tau}\|yv\|_{L^2} \sim \dot{\tau}\|y\gamma\|_{L^2} \approx \sqrt{\ln t}}.\end{aligned}$$

□

REMARK 7.3. These results remain valid when the logarithmic nonlinearity is perturbed by an energy-subcritical, defocusing powerlike nonlinearity,

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u + \mu |u|^{2\sigma} u, \quad u_{|t=0} = u_0,$$

with $\mu > 0$ and $0 < \sigma < \frac{2}{(d-2)_+}$. Surprisingly enough, the logarithmic nonlinearity is thus the stronger in the above equation.

7.2. Elements of proof.

7.2.1. *A priori estimates.* The key step is to change the unknown function in order to get coercivity. The change of unknown function is motivated by the explicit computations in the Gaussian case: at leading order, all the Gaussian solution have the same dispersion, the same oscillations, and the same asymptotic profile: Theorem 7.1 states that these three properties are shared by *all* solutions.

Direct computations show that v, given by (7.2), solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2 - \lambda d v \ln \tau + 2\lambda v \ln\left(\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\right),$$

where we recall that $\gamma(y) = e^{-|y|^2/2}$, and the initial datum for v is

$$v_{|t=0} = v_0 := \frac{\|\gamma\|_{L^2(\mathbb{R}^d)}}{\|u_0\|_{L^2(\mathbb{R}^d)}} u_0.$$

Replacing v with $ve^{-i\theta(t)}$ for

$$\theta(t) = \lambda d \int_0^t \ln \tau(s) ds - 2\lambda t \ln(\|u_0\|_{L^2}/\|\gamma\|_{L^2}),$$

we may assume that the last two terms are absent, and we focus our attention on

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln \left|\frac{v}{\gamma}\right|^2, \quad v_{|t=0} = v_0 \,. \tag{7.3}$$

The above equation is still Hamiltonian: introduce

$$\mathcal{E}(t) := \operatorname{Im} \int_{\mathbb{R}^d} \bar{v}(t,y)\partial_t v(t,y)dy = \mathcal{E}_{\rm kin}(t) + \lambda \mathcal{E}_{\rm ent}(t)\,,$$

where

$$\mathcal{E}_{\rm kin}(t) := \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2$$

is the (modified) kinetic energy and

$$\mathcal{E}_{\rm ent}(t) := \int_{\mathbb{R}^d} |v(t,y)|^2 \ln \left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy$$

is a relative entropy. Direct computations yield

$$\dot{\mathcal{E}} = -2\frac{\dot\tau}{\tau}\mathcal{E}_{\rm kin}\,. \tag{7.4}$$

Lemma 7.4. *Under the assumptions of Theorem 7.1,*

$$\sup_{t\geqslant 0}\left(\int_{\mathbb{R}^d}\left(1+|y|^2+\left|\ln|v(t,y)|^2\right|\right)|v(t,y)|^2dy + \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)}\right) < \infty$$

and

$$\int_0^\infty \frac{\dot\tau(t)}{\tau^3(t)}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)}dt < \infty. \tag{7.5}$$

Proof. Write the pseudo-energy $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$, where $\mathcal{E}_+$ gathers the positive terms of $\mathcal{E}$,

$$\mathcal{E}_+(t) = \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2 + \lambda\int_{|v|>1}|v|^2\ln|v|^2 + \lambda\int_{\mathbb{R}^d}|y|^2|v|^2,$$

and

$$\mathcal{E}_-(t) = \lambda\int_{|v|<1}|v|^2\ln|v|^2 \leqslant 0.$$

Since $\mathcal{E}$ is nonincreasing,

$$\mathcal{E}_+(t) \leqslant \mathcal{E}(0) - \mathcal{E}_-(t) \leqslant \mathcal{E}(0) + C_\varepsilon\int_{|v|<1}|v|^{2-\varepsilon} \leqslant \mathcal{E}(0) + C_\varepsilon\int_{\mathbb{R}^d}|v|^{2-\varepsilon},$$

for any $0<\varepsilon<2$. Considering $0<\varepsilon<\frac{4}{d+2}$, we have

$$\int_{\mathbb{R}^d}|v|^{2-\epsilon} \lesssim \|v\|_{L^2}^{2-(1+d/2)\epsilon}\|yv\|_{L^2}^{d\epsilon/2}.$$

Noting that $\|v(t)\|_{L^2} = \|v(0)\|_{L^2} (= \|\gamma\|_{L^2})$, we obtain a control of the form

$$\mathcal{E}_+(t) \leqslant \mathcal{E}(0) + C\mathcal{E}_+(t)^{d\epsilon/2},$$

hence $\mathcal{E}_+ \in L^\infty(\mathbb{R}_+)$ by picking $\varepsilon > 0$ sufficiently small. Then $\mathcal{E}_- \in L^\infty(\mathbb{R}_+)$, hence $\mathcal{E} \in L^\infty(\mathbb{R}_+)$, and (7.5) by just saying that $\dot{\mathcal{E}}$ is integrable. □

7.2.2. *Center of mass.* Adapting the computation of [**20**], introduce

$$I_1(t) := \operatorname{Im} \int_{\mathbb{R}^d} \overline{v}(t,y)\nabla_y v(t,y)dy\,, \quad I_2(t) := \int_{\mathbb{R}^d} y|v(t,y)|^2 dy\,.$$

We compute:

$$\dot{I}_1 = -2\lambda I_2\,, \qquad \dot{I}_2 = \frac{1}{\tau(t)^2} I_1\,.$$

Set $\tilde{I}_2 = \tau I_2$: $\ddot{\tilde{I}}_2 = 0$, hence

$$I_2(t) = \frac{1}{\tau(t)}\left(\dot{\tilde{I}}_2(0)t + \tilde{I}_2(0)\right) = \frac{1}{\tau(t)}\left(-I_1(0)t + I_2(0)\right) = \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right).$$

In particular, $\displaystyle\int_{\mathbb{R}^d} y|v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} 0 = \int_{\mathbb{R}^d} y\gamma(y)^2 dy$. If $I_1(0) \neq 0$, we also have

$$I_1(t) \underset{t\to\infty}{\sim} c\frac{t}{\sqrt{\ln t}},$$

while if $I_1(0) = 0 \neq I_2(0)$,

$$I_1(t) \underset{t\to\infty}{\sim} \tilde{c}\sqrt{\ln t}.$$

Remark 7.5. In view of Cauchy-Schwarz inequality,

$$|I_1(t)| \leqslant \|v\|_{L^2}\|\nabla_y v\|_{L^2} = \|\gamma\|_{L^2}\|\nabla_y v\|_{L^2}.$$

So unless the initial data are centered in zero in phase space ($I_1(0) = I_2(0) = 0$),

$$\|\nabla_y v\|_{L^2} \underset{t\to\infty}{\longrightarrow} \infty,$$

suggesting that v is rapidly oscillatory: in general, (7.2) filters out the *leading order* oscillations only, in the limit $t \to \infty$.

7.2.3. *Second order momentum.* Introduce $J = \operatorname{Im}\displaystyle\int v\, y\cdot\nabla_y \bar{v}$. Cauchy-Schwarz: $|J| \leqslant \|yv\|_{L^2}\|\nabla v\|_{L^2} \lesssim \tau(t)$ (previous lemma).

Use the conservation of the energy of u:

$$\frac{d}{dt}\Bigg(E_{\text{kin}} + \frac{(\dot\tau)^2}{2}\int |y|^2|v|^2 - \frac{\dot\tau}{\tau}J + \lambda\int |v|^2\ln|v|^2 - \lambda d\ln\tau\int |v|^2$$
$$+ 2\lambda\|\gamma\|_{L^2}^2 \ln\left(\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}\right)\Bigg) = 0.$$

$$\frac{(\dot\tau)^2}{2}\int |y|^2|v|^2 - \lambda d\ln\tau\int |v|^2 = \mathcal{O}(\dot\tau).$$

But $(\dot\tau)^2 = 2\lambda\ln\tau$ and $\|v\|_{L^2}^2 = \|\gamma\|_{L^2}^2 = \frac{2}{d}\|y\gamma\|_{L^2}^2$.

$$\|yv(t)\|_{L^2}^2 - \|y\gamma\|_{L^2}^2 = \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right).$$

7.2.4. *Universal profile.* Madelung:

- Formal: $v = \sqrt{\rho}e^{i\phi}$. Vacuum...
- Rigorous: $\rho = |v|^2$, $J = \operatorname{Im} \bar{v}\nabla v$.

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0, \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = \dfrac{1}{4\tau^2}\Delta\nabla\rho - \dfrac{1}{\tau^2}\nabla \cdot \operatorname{Re}\left(\nabla v \otimes \nabla \bar{v}\right). \end{cases}$$

Baby model:

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0, \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = 0. \end{cases}$$

In terms of ρ only: $\partial_t \left(\tau^2 \partial_t \rho\right) = \lambda \nabla \cdot (\nabla + 2y)\,\rho =: \lambda L\rho$.

Note that $\tau^2 \ll (\dot{\tau}\tau)^2$: define s such that $\dfrac{\dot{\tau}\tau}{\lambda}\partial_t = \partial_s$,

$$s = \int \frac{\lambda}{\dot{\tau}\tau} = \int \frac{\ddot{\tau}}{2\dot{\tau}} = \frac{1}{2}\ln \dot{\tau}(t)\,.$$

Notice that

$$s \sim \frac{1}{4}\ln\ln t\,, \quad t \to \infty\,.$$

Then again discarding formally lower order terms we find

$$\partial_s \rho = L\rho.$$

Remark 7.6. Recall that $\rho(t,y) = |v(t,y)|^2$: logarithmic convergence in time,

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \rho(t,y)dy = \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y)dy + \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right).$$

We have derived formally:

$$\partial_s \rho = L\rho, \quad L = \nabla \cdot (\nabla + 2y)\,.$$

For such Fokker–Planck equation, convergence to equilibrium with an (spectral gap),

$$\|\rho(s) - \gamma^2\|_{L^1} \lesssim e^{-Cs}\|\rho_0 - \gamma^2\|_{L^1}.$$

Both aspects coincide, since

$$s \sim \frac{1}{4}\ln\ln t\,, \quad t \to \infty\,.$$

Back to the complete the hydrodynamical system: Eliminate j, and introduce the time variable s, $\tilde{\rho}(s,y) := \rho(t,y)$:

$$\partial_s \tilde{\rho} - \frac{2\lambda}{(\dot{\tau})^2}\partial_s \tilde{\rho} + \frac{\lambda}{(\dot{\tau})^2}\partial_s^2 \tilde{\rho} = L\tilde{\rho} - \frac{1}{4\lambda\tau^2}\Delta^2\tilde{\rho} - \frac{1}{\tau^2}\nabla \cdot \nabla \cdot \operatorname{Re}\left(\nabla v \otimes \nabla \bar{v}\right).$$

For $s \in [-1,2]$ and $s_n \to \infty$, set $\tilde{\rho}_n(s,y) = \tilde{\rho}(s+s_n,y)$.
De la Vallée-Poussin+ Dunford–Pettis yields (up to a subsequence)

$$\tilde{\rho}_n \rightharpoonup \tilde{\rho}_\infty \text{ in } L^p_s(-1,2;L^1_y), \quad \forall p \in [1,\infty).$$

$$\partial_s \tilde{\rho}_\infty = L\tilde{\rho}_\infty \text{ in } \mathcal{D}'\left((-1,2)\times\mathbb{R}^d\right).$$

Since $J = \operatorname{Im} \bar{v}\nabla_y v$,Lemma 7.4 yields

$$\frac{\dot{\tau}}{\tau}\tilde{J} \in L^2_s L^1_y, \quad \text{hence } \frac{\dot{\tau}}{\tau}\nabla\cdot\tilde{J}_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1}).$$

Therefore, $\partial_s \tilde{\rho}_\infty = 0$.

It is known from [**3**] that any solution to

$$\partial_s \tilde{\rho}_\infty = L\tilde{\rho}_\infty$$

satisfying the a priori estimates of Lemma 7.4 converges for large time

$$\lim_{s\to\infty} \|\tilde{\rho}_\infty(s) - \gamma^2\|_{L^1(\mathbb{R}^d)} = 0.$$

On the other hand, the Liouville property yields $\partial_s\tilde{\rho}_\infty = 0$, hence $\tilde{\rho}_\infty = \gamma^2$. Thus, the limit is unique, and no extraction is needed:

$$\tilde{\rho}(s) \underset{s\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d).$$

Remark 7.7. Some information is lost when approximating the original hydrodynamical system by a Fokker-Planck equation: this is the reason why only a weak convergence is obtained. This should not be too surprising, as the Fokker-Planck equation is parabolic, while we started from a Hamiltonian equation. On the other hand, in [**23**], by changing the strategy of proof, the convergence is improved: Denoting by W_1 the Wasserstein distance, there exists C such that

$$W_1\left(\frac{|v(t)|^2}{\pi^{d/2}}, \frac{\gamma^2}{\pi^{d/2}}\right) \leqslant \frac{C}{\sqrt{\ln t}}, \quad t \geqslant e.$$

For ν_1 and ν_2 probability measures,

$$W_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\mu(x,y)\right)^{1/p}; \quad (\pi_j)_\sharp\mu = \nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$, and $\pi_j : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ denotes the canonical projection onto the j-th factor. See e.g. [**34**]. In the case $p = 1$,

$$W_1(\nu_1,\nu_2) = \sup\left\{\int_{\mathbb{R}^d} \Phi d(\mu_1-\mu_2),\ \Phi\in C(\mathbb{R}^d;\mathbb{R}),\ \operatorname{Lip}(\Phi)\leqslant 1\right\},$$

Part 2. Isothermal fluids

8. From NLS to compressible fluids

Consider the solution u to (2.1), and resume the Madelung transform $\rho = |u|^2$, $j = \operatorname{Im}\bar{u}\nabla u$. The unknown (ρ, j) solves the Korteweg system:

$$\begin{cases} \partial_t\rho + \nabla\cdot j = 0, \\ \partial_t j + \nabla\left(\dfrac{j\otimes j}{\rho}\right) + \nabla\left(\rho^\gamma\right) = \dfrac{1}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases}$$

with

$$\lambda = \frac{\gamma}{\gamma-1}, \quad \sigma = \frac{\gamma-1}{2}.$$

The capillarity term (RHS of the second equation), involving the term $\frac{1}{2}\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$ also known as quantum pressure or Bohm potential in quantum mechanics, can be written in several fashions, e.g.:

$$\begin{aligned}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) &= \frac{1}{2}\nabla\cdot\left(\rho\nabla^2\ln\rho\right) = \nabla\cdot\left(\sqrt{\rho}\nabla^2\sqrt{\rho}-\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\right)\\ &= \frac{1}{2}\nabla\Delta\rho - 2\nabla\cdot\left(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\right).\end{aligned}$$

See for instance [**1, 15**]. Either of these formulas may be used, typically when constructing solutions to the Korteweg equation, according to the level of regularity considered, and the presence or absence of vacuum.
$\gamma > 1$: polytropic fluid.
$\gamma = 1$: isothermal fluid.

9. The limit $\gamma \to 1$

Fluid side: "clear". NLS side: "$|u|^{2\sigma}u \to \ln(|u|^2)u$ as $\sigma \to 0$".

[**35**]: the ground state of

$$-\frac{1}{2}\Delta\phi + \omega\phi = |\phi|^{2\sigma}\phi$$

converges, as $\sigma \to 0$, to the ground state of

$$-\frac{1}{2}\Delta\phi + \omega\phi = \phi\ln|\phi|,$$

that is, the Gausson (up to invariants).

Apart from this very specific case, it is difficult to give a rigorous meaning to the limit $\gamma \to 1$, or even construct solutions the case $\gamma = 1$. In the case of (2.1), we have seen that the (nonlinear) potential energy is

$$\frac{\lambda}{\sigma+1}\int_{\mathbb{R}^d}|u(t,x)|^{2\sigma+2}dx,$$

and becomes, in the case of (3.1),

$$\lambda\int_{\mathbb{R}^d}|u(t,x)|^2\left(\ln|u(t,x)|^2-1\right)dx.$$

It is no longer sign definite. In the fluid case, using the conservation of mass, the standard entropy in the isothermal case reads

$$\int_{\mathbb{R}^d}\rho(t,x)\ln\rho(t,x)dx,$$

and we naturally face the same property. There is however a major difference regarding the Cauchy problem: (2.1) is semilinear (for $\sigma < \frac{2}{(d-2)_+}$ it is solved in $H^1(\mathbb{R}^d)$ by using a fixed point argument, and the nonlinearity is viewed as a perturbation, see e.g. [**18**]), while the above Korteweg equation is quasilinear (nonlinear terms cannot be viewed as perturbations in general). The Cauchy problem is in general still a major issue for the equations of compressible fluid mechanics which we now discuss, in the sense that the optimal assumptions to construct weak solutions are not always known; see e.g. [**31**] and references therein. For this reason, we distinguish rigidity results ("if theorem") and the construction of weak solutions.

On the other hand, the presence of a pressure term of isothermal form in the large time limit can be guessed as follows. Consider more generally a barotropic (convex) pressure law $P(\rho)$, not necessarily equal to ρ^γ. Since the gradient of the pressure is involved, the value of $P(0)$ is irrelevant from a mathematical point of view, and we assume $P(0) = 0$. If the density ρ is dispersive in the large time limit, then the Taylor expansion of P at zero determines the large time behavior:

$$P(\rho) \underset{\rho\to 0}{\sim} P'(0)\rho + \frac{1}{2}P''(0)\rho^2 + \dots$$

If $P'(0) > 0$, then isothermal effects are present at leading order, while if $P'(0) = 0$, the dynamics corresponds to polytropic fluids. This is another way, probably more natural, to interpret Remark 7.3.

10. Setting

From now on, we no longer write any Schrödingerogner equation, and u denotes the fluid velocity, whose rigorous definition requires some care (an issue which we do not address here), and which corresponds to the momentum divided by the density,

$$u = \frac{j}{\rho},$$

outside of vacuum, that is for $\rho > 0$ ($\rho \geqslant 0$ in general). We consider

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho = \dfrac{\varepsilon^2}{2}\rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) + \nu \nabla \cdot (\rho \mathbb{D} u), \end{cases} \tag{10.1}$$

with a capillarity $\varepsilon \geqslant 0$, a viscosity $\nu \geqslant 0$, and where $\mathbb{D}u = \frac{1}{2}(\nabla u + \nabla u^\top)$ denotes the symmetric part of ∇u. The first term of the RHS corresponds to capillarity (Korteweg term), and the second is a quantum Navier-Stokes correction, see [**11**]: contrary to the Newtonian case involving $\nu \Delta u$ (see e.g. [**21, 29**]), the viscosity can be thought of as linear in ρ; see [**9, 10**] for more general models and their analysis.

We shall not detail here the notion of solution adopted in [**13, 12**], and present the main results or ideas in a rather superficial way.

Formally, the mass is conserved in (10.1),

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho(t,x)dx = 0,$$

and the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \rho|u|^2 dx + \frac{\varepsilon^2}{2}\int |\nabla\sqrt{\rho}|^2 dx + \int_{\mathbb{R}^d} \rho \ln \rho \, dx, \tag{10.2}$$

satisfies

$$\dot{E}(t) = -\nu \int_{\mathbb{R}^d} \rho |\mathbb{D}u|^2 dx. \tag{10.3}$$

We do not write the dependence of the integrated functions upon (t, x) to shorten notations.

11. Rigidity in isothermal fluids

The end of the proof of Theorem 7.1 relies on a hydrodynamical approach, suggesting that some results remain valid if we start from the isothermal Korteweg equation. The argument presented in Section 7.2.4 suggests that the capillary term has no influence in the large time behavior at leading order: assuming $\varepsilon > 0$ or $\varepsilon = 0$ in (10.1) is not expected to change the large time description. More surprisingly, the presence of the quantum Navier-Stokes correction has no influence either: we may suppose $\nu = 0$ or $\nu > 0$.

In view of (7.2) and Madelung transform, we change the unknown functions (ρ, u) to (R, U) through the relations

$$\rho(t,x) = \frac{1}{\tau(t)^d} R\left(t, \frac{x}{\tau(t)}\right) \frac{\|\rho_0\|_{L^1}}{\|\Gamma\|_{L^1}}, \quad u(t,x) = \frac{1}{\tau(t)} U\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot\tau(t)}{\tau(t)} x, \tag{11.1}$$

where we denote by y the spatial variable for R and U. The function τ is the same as in Theorem 7.1, given by (7.1). The function Γ is defined by $\Gamma(y) = e^{-|y|^2}$; in other words, $\Gamma = \gamma^2$ as defined in Theorem 7.1. The system (10.1) becomes, in terms of these new unknowns,

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2} \nabla \cdot (RU) = 0, \\ \partial_t (RU) + \dfrac{1}{\tau^2} \nabla \cdot RU \otimes U) + 2\kappa y R + \nabla R \\ \qquad\qquad = \dfrac{\varepsilon^2}{2\tau^2} R \nabla \left(\dfrac{\Delta \sqrt{R}}{\sqrt{R}} \right) + \dfrac{\nu}{\tau^2} \nabla \cdot (R \mathbb{D} U) + \nu \dfrac{\dot\tau}{\tau} \nabla R. \end{cases} \tag{11.2}$$

We define the pseudo-energy $\mathcal{E}$ of the system (11.2) by

$$\mathcal{E}(t) := \frac{1}{2\tau^2} \int R|U|^2 + \frac{\varepsilon^2}{2\tau^2} \int |\nabla \sqrt{R}|^2 + \int (R|y|^2 + R \ln R), \tag{11.3}$$

which formally satisfies

$$\dot{\mathcal{E}}(t) = -\mathcal{D}(t) - \nu \frac{\dot\tau(t)}{\tau(t)^3} \int R(t,y) \nabla \cdot U(t,y) dy, \tag{11.4}$$

where the dissipation $\mathcal{D}(t)$ is defined by

$$\mathcal{D}(t) := \frac{\dot\tau}{\tau^3} \int R|U|^2 + \varepsilon^2 \frac{\dot\tau}{\tau^3} \int |\nabla \sqrt{R}|^2 + \frac{\nu}{\tau^4} \int R |\mathbb{D} U|^2. \tag{11.5}$$

Mimicking the proof of Lemma 7.4, it is natural to expect that each term in $\mathcal{E}$ is bounded (recall that $\mathcal{E}$ is not signed, because of the logarithm), and that $\dot{\mathcal{E}}$ is integrable. This is formally a natural assumption, but as the Cauchy problem is a delicate issue, the following result remains an "if theorem" in most cases.

THEOREM 11.1 ([**13**]). *Let $\varepsilon, \nu \geqslant 0$, and let (R, U) be a global weak solution of* (11.2).

(1) *If $\int_0^\infty \mathcal{D}(t)\,\mathrm{d}t < \infty$, then*

$$\int_{\mathbb{R}^d} y R(t,y) \mathrm{d}y \underset{t\to\infty}{\longrightarrow} 0 \quad \textit{and} \quad \left| \int_{\mathbb{R}^d} (RU)(t,y) \mathrm{d}y \right| \underset{t\to\infty}{\longrightarrow} \infty,$$

unless $\int yR(0,y)\mathrm{d}y = \int (RU)(0,y)\mathrm{d}y = 0$*, a case where*

$$\int_{\mathbb{R}^d} yR(t,y)\mathrm{d}y = \int_{\mathbb{R}^d} (RU)(t,y)\mathrm{d}y \equiv 0.$$

(2) *If* $\sup_{t\geqslant 0} \mathcal{E}(t) + \int_0^\infty \mathcal{D}(t)\,\mathrm{d}t < \infty$*, then* $R(t,\cdot) \rightharpoonup \Gamma$ *weakly in* $L^1(\mathbb{R}^d)$ *as* $t\to\infty$*.*

(3) *If* $\sup_{t\geqslant 0} \mathcal{E}(t) < \infty$ *and the energy* E *defined by* (10.2) *satisfies* $E(t) = o(\ln t)$ *as* $t\to\infty$*, then*

$$\int_{\mathbb{R}^d} |y|^2 R(t,y)\mathrm{d}y \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} |y|^2 \Gamma(y)\mathrm{d}y.$$

Essentially, the proof is based on arguments similar to those sketched in Section 7.2. As evoked above, it is a bit of a surprise that the Navier-Stokes term goes through the arguments, and we refer to [**13**] for details.

Remark 11.2. In the same spirit as the discussion at the end of Section 9, the pressure law considered in [**13**] is more general than exactly isothermal: we assume that $P \in C^1([0,\infty[;\mathbb{R}_+)\cap C^2(]0,\infty[;\mathbb{R}_+)$, and P is convex, with $P'(0)>0$.

12. Existence

As already evoked, constructing solutions in compressible fluid mechanics is a difficult question.

(...)

In [**12**], we construct weak solutions to (10.1) in the presence of viscosity, $\nu > 0$. We emphasize two aspects in this construction, which seem to be the more important contributions of this work:

- We consider solutions on the whole space $\mathbb{R}^d$, while most of the previous references assume a periodic setting, $x \in \mathbb{T}^d$.
- We gain positivity properties by working on the intermediary system (11.2).

Both points are intimately connected, as the change of unknown functions (11.1) involves a time-dependent rescaling. The reasons why most of the references consider the periodic setting $x\in\mathbb{T}^d$ seem to be mostly that compactness in space then comes from free, and integrations by parts can be performed freely. The periodic case is also rather convenient for approximating, among others in Lebesgue spaces, the initial density by a density bounded away from zero, a step which would require some modification on $\mathbb{R}^d$. Note also that this property is classically propagated by the flow in a suitable regularized continuity equation (see e.g. [**21, 28**]), and such a property is needed in the presence of cold pressure and regularizing terms (see e.g. [**25, 33**]).

For these reasons, to construct a solution (R,U) to (11.2) on $\mathbb{R}^d$, we first replace $\mathbb{R}^d$ with a box $\mathbb{T}^d_\ell$ of size $\ell>0$, where ℓ is aimed at going to infinity at the last step of the proof.

We refer to [**12**] for the details, and conclude this section by pointing out another important tool, which has proven very useful in the context of compressible

Navier-Stokes with a density-dependent velocity, known as BD-entropy, after [**7, 8**]. It involves an effective velocity, which reads $U+\nu\nabla\ln R$ in the case of (11.2):

$$\mathcal{E}_{\rm BD}(R,U)=\frac{1}{2\tau^2}\int_{\mathbb{R}^d}\left(R|U+\nu\nabla\log R|^2+\varepsilon^2|\nabla\sqrt{R}|^2\right)+\int_{\mathbb{R}^d}\left(R|y|^2+R\log R\right).$$

The evolution of this BD-entropy is given formally, for $t\geqslant 0$, by

$$\begin{aligned}&\mathcal{E}_{\rm BD}(R,U)(t)+\int_0^t\mathcal{D}_{\rm BD}(R,U)(s)\mathrm{d}s\\&\qquad=\mathcal{E}_{\rm BD}(R_0,U_0)+\nu\int_0^t\frac{2d}{\tau^2}\int_{\mathbb{R}^d}R+\nu\int_0^t\frac{\dot\tau}{\tau^3}\int_{\mathbb{R}^d}R\nabla\cdot U,\end{aligned}\tag{12.1}$$

where the above dissipation is defined by

$$\begin{aligned}\mathcal{D}_{\rm BD}(R,U)=\frac{\dot\tau}{\tau^3}&\int\left(R|U|^2+\varepsilon^2|\nabla\sqrt{R}|^2\right)+\frac{\nu}{\tau^4}\int_{\mathbb{R}^d}R|\mathbb{A}U|^2\\&+\frac{\nu\varepsilon^2}{\tau^4}\int R|\nabla^2\log R|^2+\frac{4\nu}{\tau^2}\int|\nabla\sqrt{R}|^2,\end{aligned}\tag{12.2}$$

with $\mathbb{A}U:=\frac12(\nabla U-\nabla U^\top)$ the skew-symmetric part of ∇U. Hence putting together the energy and the BD-entropy equalities, it holds

$$\mathcal{E}(t)+\mathcal{E}_{\rm BD}(t)+\int_0^t\left(\mathcal{D}(s)+\mathcal{D}_{\rm BD}(s)\right)ds=\mathcal{E}(0)+\mathcal{E}_{\rm BD}(0)+\nu\int_0^t\frac{2d}{\tau^2}\int_{\mathbb{R}^d}R,\quad t\geqslant 0,$$

and thanks to the conservation of mass and the fact that $\int_0^\infty\tau^{-2}(t)\,\mathrm{d}t<\infty$, the last term is uniformly bounded.

Theorem 12.1 ([**12**]). *Assume $\nu>0$, $\varepsilon\geqslant 0$. Let $(\sqrt{R_0},\Lambda_0=(\sqrt{R}U)_0)\in L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ satisfy $\mathcal{E}(0)<\infty$, $\mathcal{E}_{\rm BD}(0)<\infty$, as well as the compatibility conditions*

$$\sqrt{R_0}\geqslant 0\ \text{a.e. on }\mathbb{R}^d,\quad(\sqrt{R}U)_0=0\ \text{a.e. on }\{\sqrt{R_0}=0\}.$$

There exists at least one global weak solution to (11.2)*, which satisfies moreover the energy and BD-entropy inequalities: There exist absolute constants C,C' such that, for almost all $t\geqslant 0$, there holds:*

$$\mathcal{E}(t)+\int_0^t\mathcal{D}(s)\,\mathrm{d}s\leqslant C(\mathcal{E}(0)),\tag{12.3}$$

$$\mathcal{E}_{\rm BD}(t)+\int_0^t\mathcal{D}_{\rm BD}(s)\,\mathrm{d}s\leqslant C'(\mathcal{E}(0),\mathcal{E}_{\rm BD}(0)),\tag{12.4}$$

with $\mathcal{E},\mathcal{D},\mathcal{E}_{\rm BD},\mathcal{D}_{\rm BD}$ as defined in (11.3)-(11.4)-(12.1)-(12.2).

This result implies existence results for (10.1), see [**12**]

13. From isothermal to polytropic

The method of proof developed to study (3.1) and (10.1) turns out be bring some information in the case of (2.1) and polytropic fluids, as shown in [**14**]. Replace (7.1) with

$$\ddot\tau=\frac{\alpha}{2\tau^{1+\alpha}},\quad\tau(0)=1,\ \dot\tau(0)=0.\tag{13.1}$$

Its large time behavior turns out to be independent of $\alpha>0$:

LEMMA 13.1. *Let $\alpha > 0$. The ordinary differential equation* (13.1) *has a unique, global, smooth solution $\tau \in C^\infty(\mathbb{R};\mathbb{R}_+)$. In addition, its large time behavior is given by*

$$\dot{\tau}(t) \underset{t\to\infty}{\longrightarrow} 1, \quad \textit{hence } \tau(t) \underset{t\to\infty}{\sim} t.$$

We see that the value of the parameter $\alpha > 0$ does not influence the large time behavior, at leading order. And in view of Theorem 7.1, the behavior changes for $\alpha = 0$, by a logarithmic factor (which turns out to be the key of e.g. Corollary 7.2). All the algebra presented so far can then be resumed: we change unknown functions as in (7.2) and (11.1), and obtain equations analogous to (7.3) and (11.2). The choice of α is suggested by the value of σ (or, equivalently, γ). Informally, the main result for fluid dynamics in [**14**] is again an "if theorem", as in [**13**]: every solution to the analogue of (11.2), where, among others, ∇R is replaced by ∇R^γ, satisfying suitable conditions, has an asymptotic profile, that is, there exists $R_\infty \in \mathbb{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$, with two finite momenta, such that

$$R(t,\cdot) \rightharpoonup R_\infty \quad \text{in } \mathbb{P}(\mathbb{R}^d), \quad \text{as } t \to \infty.$$

We have in addition $R_\infty \in L^1(\mathbb{R}^d)$ (at least) in the following cases:

- $\varepsilon = \nu = 0$ and $1 < \gamma \leqslant 1 + 2/d$,
- $\varepsilon > 0$, $\nu = 0$ and $\gamma > 1$,
- $\varepsilon \geqslant 0$, $\nu > 0$ and $1 < \gamma \leqslant 1 + 1/d$.

The results of [**32, 26**] in the case of the Euler equation ($\varepsilon = \nu = 0$) and the scattering results for the nonlinear Schrödinger equation (for the Korteweg equation $\varepsilon > 0 = \nu$) show that unlike what has been established in the isothermal case, the profile R_∞ is not universal.

References

[1] P. Antonelli and P. Marcati. On the finite energy weak solutions to a system in quantum fluid dynamics. *Comm. Math. Phys.*, 287(2):657–686, 2009.

[2] A. H. Ardila. Orbital stability of Gausson solutions to logarithmic Schrödinger equations. *Electron. J. Differential Equations*, pages Paper No. 335, 9, 2016.

[3] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100, 2001.

[4] W. Bao, R. Carles, C. Su, and Q. Tang. Error estimates of a regularized finite difference method for the logarithmic Schrödinger equation. *SIAM J. Numer. Anal.*, 57(2):657–680, 2019.

[5] I. Białynicki-Birula and J. Mycielski. Nonlinear wave mechanics. *Ann. Physics*, 100(1-2):62–93, 1976.

[6] I. Białynicki-Birula and J. Mycielski. Gaussons: Solitons of the logarithmic Schrödinger equation. *Special issue on solitons in physics, Phys. Scripta*, 20:539–544, 1979.

[7] D. Bresch and B. Desjardins. Quelques modèles diffusifs capillaires de type Korteweg. *Comptes Rendus Mécanique*, 332(11):881–886, 2004.

[8] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003. `https://doi.org/10.1081/PDE-120020499`.

[9] D. Bresch and P.-E. Jabin. Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math. (2)*, 188(2):577–684, 2018.

[10] D. Bresch, A. F. Vasseur, and C. Yu. Global existence of entropy-weak solutions to the compressible Navier-Stokes equations with non-linear density dependent viscosities. Preprint. Archived at `https://arxiv.org/abs/1905.02701`, 2019.

[11] S. Brull and F. Méhats. Derivation of viscous correction terms for the isothermal quantum Euler model. *ZAMM Z. Angew. Math. Mech.*, 90(3):219–230, 2010. `https://doi.org/10.1002/zamm.200900297`.

[12] R. Carles, K. Carrapatoso, and M. Hillairet. Global weak solutions for quantum isothermal fluids. *Ann. Inst. Fourier (Grenoble)*. To appear. Archived at `https://hal.archives-ouvertes.fr/hal-02116596`.

[13] R. Carles, K. Carrapatoso, and M. Hillairet. Rigidity results in generalized isothermal fluids. *Annales Henri Lebesgue*, 1:47–85, 2018.

[14] R. Carles, K. Carrapatoso, and M. Hillairet. Large-time behavior of compressible polytropic fluids and nonlinear Schrödinger equation, 2021. Preprint, archived at `https://hal.archives-ouvertes.fr/hal-03142668`.

[15] R. Carles, R. Danchin, and J.-C. Saut. Madelung, Gross-Pitaevskii and Korteweg. *Nonlinearity*, 25(10):2843–2873, 2012.

[16] R. Carles and I. Gallagher. Universal dynamics for the defocusing logarithmic Schrödinger equation. *Duke Math. J.*, 167(9):1761–1801, 2018.

[17] T. Cazenave. Stable solutions of the logarithmic Schrödinger equation. *Nonlinear Anal.*, 7(10):1127–1140, 1983.

[18] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[19] T. Cazenave and A. Haraux. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Sci. Toulouse Math. (5)*, 2(1):21–51, 1980.

[20] P. Ehrenfest. Bemerkung über die angenaherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik. *Zeitschrift für Physik*, 45(7-8):455–457, 1927.

[21] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[22] G. Ferriere. The focusing logarithmic Schrödinger equation: analysis of breathers and nonlinear superposition. *Discrete Contin. Dyn. Syst.*, 40(11):6247–6274, 2020.

[23] G. Ferriere. Convergence rate in Wasserstein distance and semiclassical limit for the defocusing logarithmic Schrödinger equation. *Anal. PDE*, 14(2):617–666, 2021.

[24] G. Ferriere. Existence of multi-solitons for the focusing logarithmic non-linear Schrödinger equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 38(3):841–875, 2021.

[25] M. Gisclon and I. Lacroix-Violet. About the barotropic compressible quantum Navier-Stokes equations. *Nonlinear Anal.*, 128:106–121, 2015. `https://doi.org/10.1016/j.na.2015.07.006`.

[26] M. Grassin. Global smooth solutions to Euler equations for a perfect gas. *Indiana Univ. Math. J.*, 47(4):1397–1432, 1998. `https://doi.org/10.1512/iumj.1998.47.1608`.

[27] M. Hayashi. A note on the nonlinear Schrödinger equation in a general domain. *Nonlinear Anal.*, 173:99–122, 2018.

[28] A. Jüngel. Global weak solutions to compressible Navier-Stokes equations for quantum fluids. *SIAM J. Math. Anal.*, 42(3):1025–1045, 2010. `https://doi.org/10.1137/090776068`.

[29] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[30] Y. Martel and F. Merle. Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(6):849–864, 2006.

[31] F. Rousset. Solutions faibles de l'équation de Navier-Stokes des fluides compressibles. *Astérisque*, pages Exp. No. 1135, 565–584, 2017. Séminaire Bourbaki, Vol. 2016/17.

[32] D. Serre. Solutions classiques globales des équations d'Euler pour un fluide parfait compressible. *Ann. Inst. Fourier*, 47:139–153, 1997.

[33] A. F. Vasseur and C. Yu. Global weak solutions to the compressible quantum Navier-Stokes equations with damping. *SIAM J. Math. Anal.*, 48(2):1489–1511, 2016. `https://doi.org/10.1137/15M1013730`.

[34] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[35] Z.-Q. Wang and C. Zhang. Convergence from power-law to logarithm-law in nonlinear scalar field equations. *Arch. Ration. Mech. Anal.*, 231(1):45–61, 2019.

Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
Email address: Remi.Carles@math.cnrs.fr